Determinants

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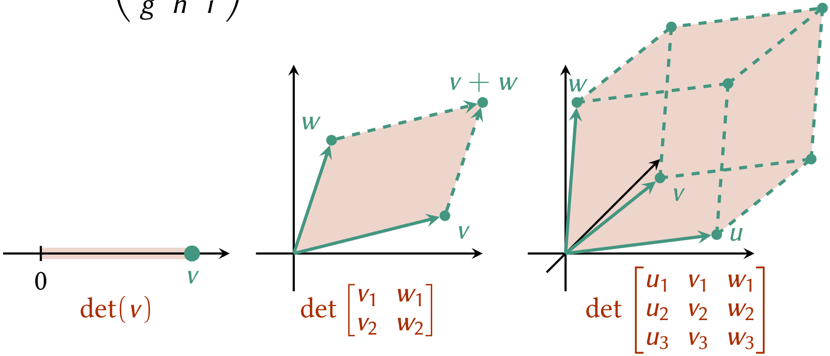
For quadratic matrices we can compute the **determinant** as

$$\det\begin{pmatrix} a \end{pmatrix} = a \qquad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = aei + bfg + cdh - gec - hfa - idb$$



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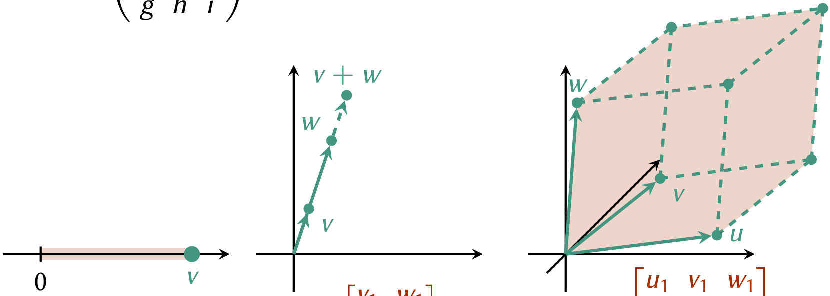

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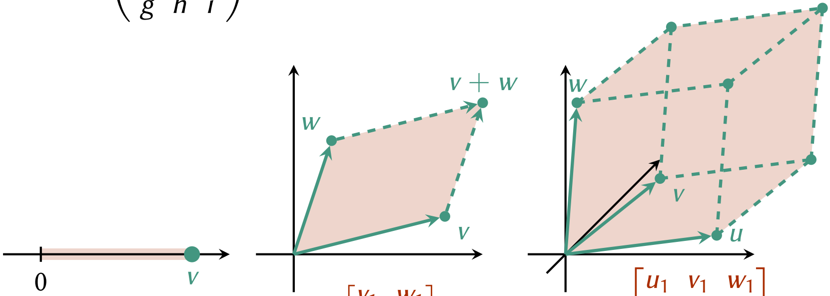

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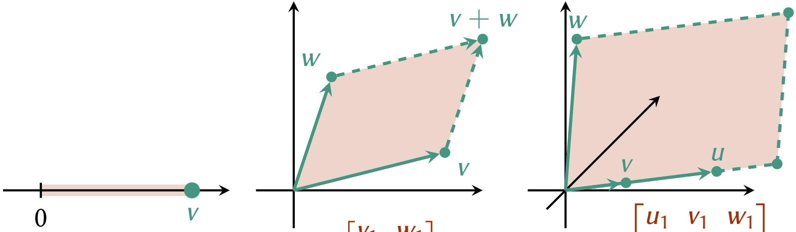

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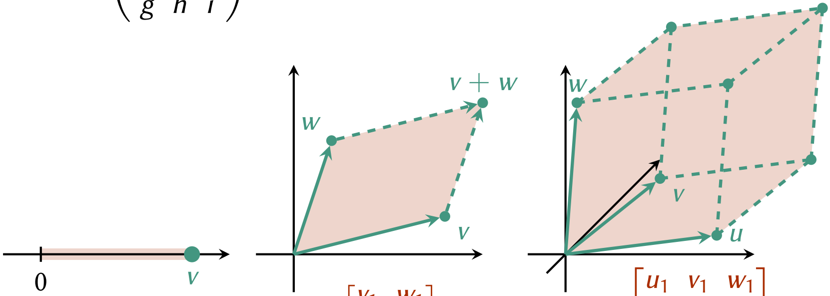

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Laplace's formula

For a quadratic matrix $A \in \mathbb{R}^{n\times n}$ for $n \geq 2$ we can compute

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij}) \qquad \text{(Expansion along row } i\text{)}$$

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij}) \qquad \text{(Expansion along column } j\text{)}$$

where

$$A_{ij} = \det \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nn} \end{pmatrix}$$

Theorem

For $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{R}$ we have

- $\det A = \det A^T$
- $\det(\lambda A) = \lambda^n \det(A)$
- $\det(AB) = \det(A)\det(B)$
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Theorem

For $A \in \mathbb{R}^{n \times n}$ the following statements are equivalent.

- $\det A \neq 0$
- *A has rank n*
- *$Ax = b$ has a unique solution*

- If A is a triangular matrix, i.e.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

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  then $\det(A) = \prod_{i=1}^{n} a_{ii}$.
- In particular if $A = \operatorname{diag}(d_1, \ldots, d_n)$ then $\det(A) = \prod_{i=1}^{n} d_i$.

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- It is possible to compute the determinant by transforming the matrix A into a triangular matrix B